Applications of the Wronskian and Gram Matrices of $\{t^i e^{\lambda_k t}\}$

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ABSTRACT

A review is made of some of the fundamental properties of the sequence of functions $\{t^i e^{\lambda_k t}\}$, $k=1,\dots,s$, $i=0,\dots,m_{k-1}$, with distinct λ_i. In particular it is shown how the Wronskian and Gram matrices of this sequence of functions appear naturally in such fields as spectral matrix theory, controllability, and Lyapunov stability theory.

1. INTRODUCTION

Let $\psi(\lambda)=\prod_{k=1}^{s}(\lambda-\lambda_k)^{m_k}$ be a complex polynomial with distinct roots $\lambda_1,\dots,\lambda_s$, and let $m=m_1+\cdots+m_s$. It is well known [9] that the functions $\{t^i e^{\lambda_k t}\}$, $k=1,\dots,s$, $i=0,1,\dots,m_{k-1}$, form a fundamental (i.e., linearly independent) solution set to the differential equation

$$\psi(\mathbb{D})y(t)=0, \tag{1}$$

where $\mathbb{D}=\dfrac{d}{dt}(\cdot)$.

It is the purpose of this review to illustrate some of the important properties of this independent solution set. In particular we shall show that both the Wronskian matrix and the Gram matrix play a dominant role in certain applications of matrix theory, such as spectral theory, controllability, and the Lyapunov stability theory. Few of the results in this paper are new; however, the proofs we give are novel and shed some new light on how the various concepts are interrelated, and on why certain results work the way they do.

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Throughout this paper, all matrices will be complex and $\mathbf{C}_{n\times n}$ denotes the set of $n\times n$ complex matrices. We shall denote the spectrum, the minimal polynomial, and a general eigenvalue of the matrix A by $\sigma(A)$, $\psi_A(\lambda)$, and $\lambda(A)$ respectively. As always, we shall denote columns by boldface letters, and use $\mathbf{e}_i^T$ to indicate the row $[0,\dots,0,1,0,\dots,0]$. We shall use $A>0$ to denote the positive definite Hermitian property, and shall say that A is stable if $\mathrm{Re}[\lambda(A)]<0$.

Furthermore we shall denote by $\mathfrak{D}[0,T]$, $T>0$, the vector space of piecewise continuous complex valued functions of a real variable t on $[0,T]$, and let $L_2[0,\infty]$ denote the vector space of square integrable functions. Both vector spaces are Hilbert spaces under the inner product

$$\langle x(t)|y(t)\rangle=\int_0^T x(t)\bar{y}(t)\,dt \tag{2}$$

with $T<\infty$ and $T=\infty$ respectively. Lastly, if $A=[\mathbf{a}_1,\mathbf{a}_2,\dots,\mathbf{a}_n]$, we shall use $\mathrm{col}(A)$ to denote $[\mathbf{a}_1^T,\mathbf{a}_2^T,\dots,\mathbf{a}_n^T]^T$, and we shall use $\otimes$ to denote the Kronecker product.

This paper consists of seven sections. In Section 1 our concepts are introduced. In Section 2 we discuss the Wronskian matrix at zero; in Section 3 the Gram matrix is introduced. In Section 4, an application of the Gram matrix to controllability is given. The spectral theorem for two commuting matrices is derived in Section 5 and applied to the Lyapunov equation in Section 6. The paper is concluded with remarks and conclusions in Section 7.

2. THE WRONSKIAN AT ZERO

Consider the differential equation (1) with fundamental solution set ordered as

$$\left(e^{\lambda_1 t},te^{\lambda_1 t},\dots,t^{m_1-1}e^{\lambda_1 t}|e^{\lambda_2 t},te^{\lambda_2 t},\dots|e^{\lambda_s t},te^{\lambda_s t},\dots,t^{m_s-1}e^{\lambda_s t}\right).$$

Then the Wronskian matrix $W(t)$ of this sequence evaluated at the origin is given by

$$W(0)=[W_1,W_2,\dots,W_s], \tag{3}$$

where

$$W_k^T=\begin{bmatrix} 1 & \lambda_k & \lambda_k^2 & \cdots & \cdots & \cdots & \lambda_k^{m-1} \\ 0 & 1 & 2\lambda_k & \cdots & \cdots & \cdots & (m-1)\lambda_k^{m-2} \\ 0 & 0 & 2 & \cdots & \cdots & \cdots & \\ & & & \ddots & & & \vdots \\ & & & & (m_k-1)! & \cdots & \frac{(m-1)!}{(m_k-1)!}\lambda_k^{m-m_k} \end{bmatrix}_{m_k\times m}, \tag{4}$$

$k=1,2,\dots,s$. Since these functions are linearly independent, we know from the theory of differential equations that $W(t)$ is invertible for all values of $t\in\mathbb{R}$, and in particular at $t=0$ [9]. The remarkable fact is that the latter is essentially the reason why the one variable spectral theorem, as given in [5, p. 104] and (7) below works.

Indeed, if $A\in\mathbb{C}_{n\times n}$ has minimal polynomial $\psi(\lambda)=\Pi_{k=1}(\lambda-\lambda_k)^{m_k}$ with $m=m_1+\cdots+m_s$ (and $\lambda_i\neq\lambda_j$), and $f(\lambda)$ is any complex valued function for which $f^{(i)}(\lambda_k)$ is defined, then one may define $f(A)$ to equal $p(A)$, where $p(\lambda)$ is any interpolation polynomial such that $p^{(i)}(\lambda_k)=f^{(i)}(\lambda_k)$. This concept is well defined, since for any two polynomials $p(\lambda)$ and $q(\lambda)$, $p(A)=q(A)\Leftrightarrow\psi|(p-q)\Leftrightarrow p^{(i)}(\lambda_k)=q^{(i)}(\lambda_k)$, $k=1,\dots,s$, $i=0,\dots,m_{k-1}$. Now consider the system of linear equations

$$\left[\mathbb{D}_\lambda^i\left(r_0+r_1\lambda+\cdots+r_{m-1}\lambda^{m-1}\right)\right]_{\lambda=\lambda_k}=f^{(i)}(\lambda_k), \tag{5}$$

$k=1,\dots,s$, $i=0,\dots,m_{k-1}$, in the variables r_i. Since the coefficient matrix is precisely the transpose $W(0)^T$ of the Wronskian matrix at 0, as given in (3), it follows that there exists a unique solution to the equation

$$W(0)^T\mathbf{r}=\mathbf{f}, \tag{6}$$

where $\mathbf{r}^T=[r_0,r_1,\dots,r_{m-1}]$ and $\mathbf{f}^T=[f(\lambda_1),f'(\lambda_1),\dots,f_{(\lambda_1)}^{(m_1-1)},\dots]$. Moreover, since the polynomial $r(\lambda)=r_0+r_1\lambda\cdots+r_{m-1}\lambda^{m-1}$, has degree $\partial r(\lambda)<m$, it follows that $r(\lambda)$ is in fact the *unique* Hermite interpolation polynomial "agreeing with $f(\lambda)$ on the spectrum of A" [5, p. 96].

The spectral theorem now simply consists of rearranging the coefficients r_i in $r(A)$ to give

$$\begin{aligned} f(A) &= [I, A, A^2, \ldots, A^{m-1}](\mathbf{r} \otimes I_m) \\ &= [I, A, \ldots, A^{m-1}]\left[\left(W^T(0)\right)^{-1}\mathbf{f} \otimes I_m\right] \\ &= [Z_1^0, Z_1^1, \ldots, Z_s^{m_s-1}](\mathbf{f} \otimes I_m) \\ &= \sum_{k=1}^{s} \sum_{i=0}^{m_k-1} f^{(i)}(\lambda_k) Z_k^i \qquad \text{(spectral theorem)}, \end{aligned} \tag{7}$$

where

$$Z = [Z_1^0, Z_1^1, \ldots, Z_s^{m_s-1}] = [I, A, \ldots, A^{m-1}]\left[\left(W^T(0)\right)^{-1} \otimes I\right]. \tag{8}$$

The latter shows not only that the Z_k^i are polynomials in A, independent of $f(\lambda)$, but also that the spectral theorem merely consists of a suitable change of basis for the algebra generated by A, namely from (A^i) to (Z_k^i).

Applying the spectral theorem (7) to $f(\lambda) = e^{\lambda t}$ yields the well-known identity

$$e^{At} = \sum_{k=1}^{s} \sum_{i=0}^{m_k-1} t^i e^{\lambda_k t} Z_k^i, \tag{9}$$

in which the Z_k^i are linearly independent. The appearance of the fundamental sequence $t^i e^{\lambda_k t}$ in (9) underscores its importance. We shall now complete the equivalence by showing that (9) in fact implies that the functions $t^i e^{\lambda_k t}$ must be linearly independent solutions to $\psi(D)y = 0$. Indeed, it follows at once from (9) that

$$\frac{d}{dt} e^{At} = A e^{At} \tag{10}$$

and hence that $0 = \psi(A)e^{At} = \psi(\mathbb{D})e^{At} = \Sigma_{k,i}\psi(\mathbb{D})t^i e^{\lambda_k t} Z_k^i$. By the independence of the Z_k^i we have that $\psi(\mathbb{D})t^i e^{\lambda_k t} = 0$. Lastly, for any polynomial $p(\lambda)$, we see from (10) that $p(A) = p(\mathbb{D})e^{At}|_{t=0} = \Sigma_{k,i}[p(\mathbb{D})t^i e^{\lambda_k t}]_{t=0} Z_k^i = \Sigma_{k,i} p^{(i)}(\lambda_k) Z_k^i$, which gives, on taking $p(\lambda) = \lambda^j$, $j = 0, \ldots, m-1$, that $[I, A, A^2, \ldots, A^{m-1}] = [Z_1^0, \ldots, Z_s^{m_s-1}]W^T(0)$. Since the Z_k^i are independent, the change of basis matrix $W(0)^T$ is invertible, and consequently if $\Sigma_{k,i}c_{ik}t^i e^{\lambda_k t}\equiv 0$ then $W(0)\mathbf{c}=\mathbf{0}\Rightarrow\mathbf{c}=\mathbf{0}$, where $\mathbf{c}^T=[c_{01},c_{11},\dots]$. In other words the $\{t^i e^{\lambda_k t}\}$ are linearly independent, as desired.

3. THE GRAM MATRIX

Let $(x_1,x_2,\dots,x_m)$ be a sequence of m vectors in a complex inner product space $(V,\langle\cdot|\cdot\rangle)$. Then it is well known [5, p. 251] that this sequence is linearly independent if and only if the Gram matrix $G(x_1,\dots,x_m)=[\langle x_i|x_j\rangle]$ is positive definite Hermitian, i.e., $G>0$. Moreover, if the sequence (x_i) is independent, we may orthogonalize this sequence to yield the orthogonal sequence $(y_1,y_2,\dots,y_m)$. This sequence is unique within a scalar multiple. In particular we may use Gram-Schmidt orthogonalization in the form

$$(y_1,y_2,\dots,y_m)=(x_1,x_1,\dots,x_m)\begin{bmatrix} t_{1s} & t_{12} & \cdots & \\ & t_{22} & & \vdots \\ & \bigcirc & \ddots & \\ & & & t_{mm}\end{bmatrix} \tag{11}$$

for some t_{ij}, or use the form

$$y_p=\left|\begin{array}{c|c} & x_1 \\ G_{p-1} & \vdots \\ & x_{p-1} \\ \hline \langle x_p|x_1\rangle\cdots\langle x_p|x_{p-1}\rangle & x_p\end{array}\right|,\qquad p=1,2,\dots,m, \tag{12}$$

where $G_p=G(x_1,\dots,x_p)$. Either series will yield the unique orthonormal sequence $(z_1,\dots,z_m)$ obtained from the (x_i).

The fact that we shall need later is that for any sequence of constants $c_1,\dots,c_m$, there exists a vector $u\in V$ such that

$$\langle x_i|u\rangle=c_i,\qquad i=1,2,\dots,m. \tag{13}$$

Indeed, if we let $u=\Sigma_{i=1}^m d_i x_i$, with d_i to be determined, then on taking inner products we obtain

$$c_1 = \langle x_1 | u \rangle = d_1 \langle x_1 | x_1 \rangle + \cdots + d_m \langle x_1 | x_m \rangle,$$

$$\vdots$$

$$c_m = \langle x_m | u \rangle = d_1 \langle x_m | x_1 \rangle + \cdots + d_m \langle x_m | x_m \rangle.$$

That is,

$$G_m \mathbf{d} = \mathbf{c}, \tag{14}$$

where $\mathbf{d}^T = [d_1, \dots, d_m]$ and $\mathbf{c}^T = [c_1, \dots, c_m]$. Hence $\mathbf{d} = G_m^{-1}\mathbf{c}$ and

$$u = \mathbf{c}^T G_m^{-1} \mathbf{x} = \begin{vmatrix} G_m & \mathbf{x} \\ \mathbf{c}^T & 0 \end{vmatrix} \frac{(-1)}{|G_m|}. \tag{15}$$

4. CONTROLLABILITY AND THE GRAM MATRIX

Let $0 \leqslant t_0 < t_f$, and let V denote the real vector space $\mathfrak{D}[t_0, t_f]$ of piecewise continuous functions on $[t_0, t_f]$. In control theory one may represent a large class of control problems by the linear system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{16}$$

where $A_{n \times n}$ and $B_{n \times r}$ are *real* and constant matrices, and $\mathbf{u}(t)$ is piecewise continuous. The system (16) is called completely controllable (C.C. for short) if for every initial state $\mathbf{x}_0$ and every final state $\mathbf{x}_f$, there exist a finite time interval $t_0 \leqslant t \leqslant t_f$ and a control vector $\mathbf{u} \in V^n$ such that (16) has a solution $\mathbf{x}(t)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t_f) = \mathbf{x}_f$.

Now since the general solution to (16) is given by

$$\mathbf{x}(t) = e^{A(t-t_0)}\mathbf{x}_0 + \int_{t_0}^{t} e^{A(t-s)} B\mathbf{u}(s)\, ds,$$

it is easily seen that the system (16) is C.C. if and only if the linear transformation $\phi: W^n \to \mathbf{C}^n$ defined by

$$\phi(\mathbf{v}(t)) = \int_0^T e^{At} B \mathbf{u}(t)\, dt \tag{17}$$

is onto, where W is the real vector space $\mathcal{D}[0,T]$ and $T = t_f - t_0$.

A fundamental result in control theory states [1, p. 84] that (16) is C.C. if and only if the controllability matrix $\mathcal{C}[A,B] =]B, AB, A^2B, \dots, A^{n-1}B]$ satisfies

$$\operatorname{rank} \mathcal{C}[A,B] = n. \tag{18}$$

That is, $\mathcal{C}[A,B]$ is of full row rank, and contains n independent columns in $\mathbf{C}^n$.

The purpose of this section is to use the Gram matrix to give a short rigorous proof of the sufficiency of this result. The necessity is easy, as may be checked in [1, p. 84]. Indeed, we start by turning $W = \mathcal{D}[0,T]$ into a complex inner product space with the usual inner product given by (2). Now select fixed p and q such that $1 \leqslant p < n$, $1 \leqslant q \leqslant r$, and consider (17) with $\mathbf{v}(t) = \bar{v}(t)\mathbf{e}_q$, $v(t) \in W$. Substituting from (9), we see that

$$\phi(\bar{v}(t)\mathbf{e}_q) = \sum_k \sum_i Z_k^i B \mathbf{e}_q \langle t^i e^{\lambda_k t} | v(t) \rangle.$$

Since the functions $\{t^i e^{\lambda_k t}\}$ are again linearly independent, we may conclude from (13) that for any choice of constants c_{ik}, $k = 1, \dots, s, i = 0, \dots, m_{k-1}$, there exists a function $v(t) \in W$ such that $\langle t^i e^{\lambda_k t} | v(t) \rangle = c_{ik}$. In particular if we select $c_{ik} = [\mathbb{D}^i(\lambda^p)]_{\lambda = \lambda_k}$, and denote this function $v(t)$ by $v_p(t)$, then we see from the spectral theorem (7) that

$$\phi(\bar{v}_p(t)\mathbf{e}_q) = \sum_k \sum_i Z_k^i (\lambda^p)_{\lambda_k}^{(i)} B \mathbf{e}_q = A^p B \mathbf{e}_q.$$

Since $A^p B \mathbf{e}_q$ is *real*, it follows that there exists a *real* piecewise continuous control vector $\mathbf{u}_{pq}(t) = \operatorname{Re}[\bar{v}_p(t)\mathbf{e}_q]$ such that

$$\phi(\mathbf{u}_{pq}(t)) = A^p B \mathbf{e}_q.$$

This means that *every* column of A^pB, $p = 0, 1, \dots, n-1$, is in the range $R(\phi)$ of ϕ. Hence if $\operatorname{rank}(\mathcal{C}[A,B]) = n$, we may conclude that there exist n independent columns in $R(\phi)$ and hence ϕ is onto, as desired. It should be remarked here that (i) our proof did not go outside the space of piecewise continuous functions, unlike some of the more established proofs [12, p. 499], which use delta functions and distributions, and from which it is not obvious that a *piecewise continuous* control function exists; (ii) the Gram matrix of the functions $\{t^i e^{\lambda_k t}\}$ was used in constructing $v_p(f)$.

5. THE SPECTRAL THEOREM IN TWO VARIABLES

Before we can turn to our last application of the Gram matrix to the Lyapunov equation $AX+XA^*=-Q$, we shall first need the spectral theorem for *two commuting* matrices. This result was first stated by Schwerdtfeger [11, p. 32]; however, this proof does not address the difficulties involved and the assumptions that have to be made. The proof we shall give is more in line with the flavor of this paper of using exponentials and fundamental sets of functions. It will be based on the identity

$$e^{sA+tB}=e^{sA}e^{tB}, \tag{19}$$

which holds whenever $AB=BA$.

Suppose $A, B \in C_{L\times L}$, with respective minimal polynomials $\psi_A(\lambda)=\prod_{k=1}^{u}(\lambda-\lambda_k)^{m_k}$, $\psi_B(\lambda)=\prod_{l=1}^{v}(\lambda-\lambda_l)^{n_l}$, with $m=m_1+\cdots+m_u$ and $n=n_1+\cdots+n_v$. Furthermore let $Z_k^i(A)$ and $Y_l^j(B)$ denote the spectral components of A and B respectively.

Now suppose $AB=BA$, and let us consider the polynomial $r(\lambda,\mu)=\sum_{p=0}^{M}\sum_{q=0}^{N}r_{pq}\lambda^p\mu^q$. Our key observation will be that

$$r(A,B)=\sum_{k=1}^{u}\sum_{i=0}^{m_u-1}\sum_{l=1}^{v}\sum_{j=1}^{n_v-1}\left[\mathbb{D}_\lambda^i\mathbb{D}_\mu^j r(\lambda,\mu)\right]_{\lambda=\lambda_k,\,\mu=\mu_l}Z_k^iY_l^j. \tag{20}$$

To prove this we note that by (19)

$$r(A,B)e^{sA+tB}=r(\mathbb{D}_s,\mathbb{D}_t)e^{sA+tB}=r(\mathbb{D}_s,\mathbb{D}_t)e^{sA}e^{tB}$$

$$=\sum_{p,q}r_{pq}\mathbb{D}_s^p\mathbb{D}_t^q\left(\sum_{k,i}s^ie^{\lambda_k s}Z_k^i\right)\sum_{j,l}\left(t^je^{\mu_l t}Y_l^j\right)$$

$$=\sum_{k,i}\sum_{l,j}\sum_{p,q}r_{pq}\mathbb{D}_s^p(s^ie^{\lambda_k s})\mathbb{D}_t^q(t^se^{\mu_l t})Z_k^iY_l^j.$$

Evaluating this at $s=0=t$, we get

$$r(A,B)=\sum_{k,i}\sum_{l,j}\sum_{p=i}\sum_{q=j} r_{pq}\frac{q!}{(p-i)!}\lambda_k^{p-i}\frac{q!}{(q-j)!}\mu_l^{q-j}Z_k^iY_l^j,$$

$$=\sum_{k,i}\sum_{j,l}\left[\mathbb{D}^i\mathbb{D}^j r(\lambda,\mu)\right]_{\lambda=\lambda_k,\,\mu=\mu_l}Z_k^iY_l^j,$$

as desired.

For any function $f(\lambda,\mu):\mathbb{C}^2\to\mathbb{C}$, we say that f *is defined* on $\sigma(A)\times\sigma(B)$ if and only if the values

$$f^{(i,j)}(\lambda_k,\mu_l)=\left[\mathbb{D}_\lambda^i\mathbb{D}_\mu^j f(\lambda,\mu)\right]_{\lambda=\lambda_k,\,\mu=\mu_l},$$

$k=1,\dots,u$, $i=0,\dots,m_{k-1}$, $l=1,\dots,v$, $j=0,\dots,n_{l-1}$, are defined and *independent* of the order in which the partial derivatives are taken. A sufficient condition for the latter to be true is that [4, p. 57] all the partial derivatives of order m_k+n_l-2 are continuous at (λ_k,μ_l).

Analogously to the one variable case, we may now uniquely define $f(A,B)$ for any function that is defined on $\sigma(A)\times\sigma(B)$, by

$$f(A,B)=r(A,B),$$

where $r(\lambda,\mu)$ is any interpolation *polynomial* that agrees with r on $\sigma(A)\times\sigma(B)$. Again this concept is well defined, since for any two polynomials $p(\lambda,\mu)$, $q(\lambda,\mu)$

$$p(A,B)=q(A,B)\quad\Leftrightarrow\quad p^{(i,j)}(\lambda_k,\mu_l)=q^{(i,j)}(\lambda_k,\mu_l).$$

Hence the spectral theorem in two variables becomes

$$f(A,B)=\sum_{k=1}^{u}\sum_{l=1}^{v}\sum_{i=0}^{m_{k-1}}\sum_{j=0}^{n_{l-1}}\left[\mathbb{D}_\lambda^i\mathbb{D}_\mu^j f(\lambda,\mu)\right]_{\lambda=\lambda_k,\,\mu=\mu_l}Z_j^iY_l^j. \tag{21}$$

Lastly, the linear system

$$\mathbb{D}_\lambda^i\mathbb{D}_\mu^j\left[\sum_{p=0}^{m-1}\sum_{q=0}^{n-1} r_{pq}\lambda^p\mu^q\right]_{\lambda=\lambda_k,\,\mu=\mu_l}=f^{(i,j)}(\lambda_k,\mu_l) \tag{22}$$

in the variables r_{pq} has as a coefficient matrix precisely $W_A^T(0)\otimes W_B^T(0)$, where $W_A(0)$ and $W_B(0)$ are Wronskian matrices of $\{s^i e^{\lambda_k t}\}$ and $\{t^j e^{\mu_l t}\}$ respectively. We may thus conclude that (22) has a unique solution, thereby establishing the existence of the unique Hermite interpolation polynomial in two variables, of degree $<m$ in λ and degree $<n$ in μ, and agreeing with f on $\sigma(A)\times\sigma(B)$.

6. THE GRAM MATRIX AND THE LYAPUNOV EQUATION

Suppose again that $A\in\mathbb{C}_{n\times n}$, with minimal polynomial $\psi(\lambda)=\prod_{k=1}^{s}(\lambda-\lambda_k)^{m_k}$ and $m=m_1+\cdots+m_k$. Let $\mathcal{L}(X)=AX+XA^*$ be the Lyapunov operator. A fundamental result states that [7, p. 270] A is stable if and only if for every $Q>0$, $\mathcal{L}(X)=-Q$ has a unique solution $X>0$.

The aim of this section is to prove sufficiency of this result by constructing a *finite* series solution [3, 6], using the spectral theorem. The Gram matrix will then be used to show that this solution is actually positive definite. The necessity of the stability of A is easy and is left as an exercise. If Z_k^i are the spectral components of A, then it is easily seen that the Z_k^{i*} and $I\otimes Z_k^i$ are the spectral components of A^* and $I\otimes A$ respectively. Now consider $AX+XA^*=-Q$ with $Q=PP^*>0$ and A stable. Taking columns we obtain $\mathcal{G}\mathbf{x}=-\mathbf{q}$, where $\mathcal{G}=I\otimes A+\bar{A}\otimes I$, $\mathbf{x}=\mathrm{col}(X)$, and $\mathbf{q}=\mathrm{col}(Q)$. Since $\lambda_k+\bar{\lambda}_l\neq 0$, $\mathcal{G}^{-1}$ exists, and because $I_n\otimes A$ and $\bar{A}\otimes I$ commute, $\mathcal{G}^{-1}$ may be computed from the spectral theorem (21) to give

$$\mathcal{G}^{-1}=\sum_{k,i}\sum_{l,j}\left[\mathbb{D}_\lambda^i\mathbb{D}_\mu^j\frac{1}{\lambda+\mu}\right]_{\lambda=\lambda_k,\,\mu=\bar{\lambda}_l}\bar{Z}_l^j\otimes Z_k^i.$$

Hence inverting $\mathbf{x}=-\mathcal{G}^{-1}\mathbf{g}$ yields

$$X=\sum_{k,i}\sum_{l,j}Z_k^i g_{k,l}^{(i,j)}PP^*Z_l^{j*},\tag{23}$$

where

$$g_{k,l}^{(i,j)}=-\left[\mathbb{D}_\lambda^i\mathbb{D}_\mu^j\frac{1}{\lambda+\mu}\right]_{\lambda=\lambda_k,\,\mu=\mu_l}.$$

Now $g_{k,l}^{(i,j)}$ may be evaluated directly as

$$g_{k,l}^{(i,j)}=\frac{(-1)^{i+j+1}(i+j)!}{(\lambda_k+\bar{\lambda}_l)^{i+j+1}}.$$

Now consider $L_2[0,\infty]$ with the inner product defined by (2). Then $\langle t^i e^{\lambda_k t} | t^j e^{\lambda_l t}\rangle = \int_0^\infty t^{i+j} e^{(\lambda_k+\lambda_l)t}\,dt = g_{k,l}^{(i,j)}$. Thus the $m\times m$ matrix $G=[g_{k,l}^{(i,j)}]$ is precisely the Gram matrix of the fundamental solution set $t^i e^{\lambda_k t}$, and hence $G>0$. Lastly, if $Z=[Z_1^0, Z_1^1,\dots, Z_s^{m_s-1}]$, then we may write (23) as

$$X=Z(I\otimes P)(G\otimes I)(I\otimes P^*)Z^*>0, \tag{24}$$

since $y^*Z=0^T \Rightarrow y=0$ by the independence of the Z_k^i. We conclude this section with the remark that (23) may of course be rewritten in the better-known form $X=\int_0^\infty e^{At}Qe^{A^*t}\,dt$.

7. REMARKS AND CONCLUSIONS

(i) It is of some interest to observe that the invertibility of the Wronskian $W(0)$ of (4) can also be proven in several other ways. For example, if $W^T(0)\mathbf{r}=\mathbf{0}$ with $\mathbf{r}^T=[r_0,r_1,\dots,r_{m-1}]$, then this means, with the aid of (5), that $[\mathbb{D}^i(r_0+r_1\lambda+\cdots+r_{m-1}\lambda^{m-1}]_{\lambda=\lambda_k}=0$. Consequently, $r(\lambda)$ is a polynomial of degree $\partial r<m$ which has m roots, including multiplicities. It then follows that $r(\lambda)\equiv 0$ and hence $\mathbf{r}=0$.

An alternative method, which is useful in numerical work, is to actually calculate $\det W(0)^T$ exactly using a Van der Monde determinant, with distinct λ_{ki}. This determinant is then evaluated using divided differences, after which m_k of the λ_{ki} are forced to approach λ_k, $k=1,\dots,s$. This gives

$$\det W(0)=\prod_{k=1}^{s}\left[1!2!\cdots(m_k-1)!\right]\prod_{1\le j<i\le s}(\lambda_i-\lambda_j)^{m_im_j}.$$

(ii) The spectral theorem (7) is actually also equivalent to the existence of the *spiked* interpolation polynomials $\phi_q^p(\lambda)$, which satisfy

$$\left[\mathbb{D}^i\phi_q^p(\lambda)\right]_{\lambda=\lambda_k}=\delta_{ip}\delta_{kq}.$$

The selection of $\mathbf{f}=\mathbf{e}_j$, $j=1,\ldots,m$, in (6) indeed guarantees the existence of such polynomials. Moreover these polynomials may be identified with the fundamental solution set to the initial value problem

$$\psi(\mathbb{D})y(t)=0, \qquad \left[y(0),y'(0),y''(0),\ldots,y^{(m-1)}(0)\right]=\mathbf{e}_j^T, \quad j=1,\ldots,m.$$

The existence of the two variable spiked interpolation polynomial follows at once from the existence of ϕ_q^p. Indeed, $\phi_q^p(\lambda)\phi_s^r(\mu)$ suffices.

(iii) Perhaps the most useful consequence of the spectral theorem in two variables is the fact that it furnishes a rigorous proof to the folklore theorem [8, p. 341; 2, p. 65] which says that in any identity between functions of two variables λ and μ, we may replace λ and μ by two commuting matrices, provided this makes "sense." To be precise, if $G(x_1,x_2)$ is a polynomial in x_1 and x_2, and $f_i(\lambda,\mu)$, $i=1,2$, are functions defined on $\sigma(A)\times\sigma(B)$, and if $g(\lambda,\mu)=G(f_1(\lambda,\mu),f_2(\lambda,\mu))$, then $g(A,B)=0$ whenever $g(\lambda,\mu)$ vanishes on $\sigma(A)\times\sigma(B)$. Indeed, let $r_i(\lambda,\mu)$ be the Hermite interpolation polynomials agreeing with $f_i(\lambda,\mu)$ on $\sigma(A)\times\sigma(B)$, $i=1,2$. Then $f_i(A,B)=r_i(A,B)$. Now if $h(\lambda,\mu)$ is the polynomial $G(r_1(\lambda,\mu),r_2(\lambda,\mu))$, then $h(A,B)=G(r_1(A,B),r_2(A,B))=g(A,B)$. The key fact now is that $h(\lambda,\mu)$ and $g(\lambda,\mu)$ *agree* on $\sigma(A)\times\sigma(B)$. This follows from a generalization of Faa di Bruno's chain rule [10] to functions of two variables, and the fact that $G(x_1,x_2)$ is a polynomial. Hence $g(A,B)=h(A,B)=0$, as desired.

(iv) The identity $e^{sA+tB}=e^{sA}e^{tB}$ is traditionally proven from the power series expansion of e^A. An alternative and equally instructive proof is to use the contour integral representation

$$e^A=\frac{1}{2\pi i}\int_\Gamma e^\lambda(\lambda I-A)^{-1}d\lambda$$

for e^A and likewise for e^B, followed by a simple application of Cauchy's theorem.

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